The Hamiltonian of the quantum trigonometric Calogero–Sutherland model in the exceptional algebra E_8

2009 J. Phys. A: Math. Theor. 42 045205

(http://iopscience.iop.org/1751-8121/42/4/045205)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 045205 (12pp) doi:10.1088/1751-8113/42/4/045205

The Hamiltonian of the quantum trigonometric Calogero–Sutherland model in the exceptional algebra E_8

J Fernández Núñez[1], **W García Fuertes**[1] **and A M Perelomov**[2]

[1] Departamento de Física, Facultad de Ciencias, Universidad de Oviedo, E-33007 Oviedo, Spain
[2] Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia

Received 11 July 2008, in final form 4 November 2008
Published 11 December 2008
Online at stacks.iop.org/JPhysA/42/045205

Abstract
We express the Hamiltonian of the quantum trigonometric Calogero–Sutherland model for the Lie algebra E_8 and coupling constant κ by using the fundamental irreducible characters of the algebra as dynamical independent variables.

PACS numbers: 02.30.Ik, 02.20.Qs, 02.20.Sv
Mathematics Subject Classification: 81R12, 17B80

1. Introduction. Basic notation

Integrable systems are important because they can be considered as zeroth-order perturbative approximations to non-integrable systems. By integrability we mean here integrability in the sense of Liouville, that is, the existence of a complete set of mutually commuting integrals of motion. During the last three decades of the past century, a plethora of highly nontrivial (classical and quantum) mechanical integrable systems was discovered; see [1, 2] for comprehensive reviews. Among these, the Calogero–Sutherland (CS) models form a distinguished class. The first analysis of a system of this kind was performed by Calogero [3], who studied, from the quantum standpoint, the dynamics on the infinite line of a set of particles interacting pairwise by rational plus quadratic potentials and found that the problem was exactly solvable. Soon afterwards, Sutherland [4] arrived to similar results for the quantum problem on the circle, this time with trigonometric interaction, and later Moser [5] proved, in terms of Lax pairs, that the classical counterparts of these models also enjoyed integrability.

The identification of the general scope of these discoveries came with the work of Olshanetsky and Perelomov [6], who realized that it is possible to associate models of this kind with all the root systems of the simple Lie algebras, and that all these models are integrable, both in the classical and the quantum framework, for interactions of the type rational (or inverse square): q^{-2}, rational plus quadratic: $q^{-2} + \omega^2 q^2$, trigonometric: $\sin^{-2} q$, hyperbolic: $\sinh^{-2} q$, and the most general, given by the Weierstrass elliptic function $\mathcal{P}(q)$.

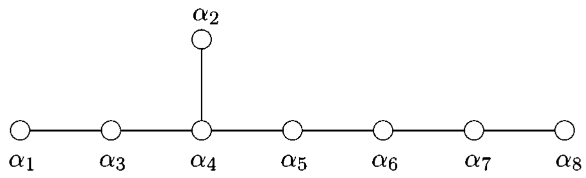

Figure 1. The Dynkin diagram for the Lie algebra E_8.

Nowadays, there is a widespread interest in this kind of integrable systems, and many applications for them have been found in a variety of fields from mathematics and physics; see for instance [7, 8]. Among the physical applications we mention the description of the boundary excitations of the fractional quantum Hall effect for high magnetic field [9], the use of the CS model to solve the DMPK equation ruling the transmission eigenvalues in disordered wires [10], the identification of the spectral curves of the CS system with the Seiberg–Witten curves which give the solution of the monodromy problem leading to the exact form of the prepotential for $N = 2$ supersymmetric Yang–Mills theory [11], and the role of the CS quantum Hamiltonian in the dynamics of the fluctuations of the configuration of D3-branes which reproduces the extreme Reissner–Nordstrom black-hole solution of the IIB superstring theory compactified on a sixth-dimensional torus [12].

As the Hamiltonian H of these systems is invariant under the Weyl group of the underlying Lie algebra, the main point of our approach in this note is to express H in a suitable set of independent variables, indeed the fundamental characters z of the underlying algebra. The use of such kind of variables has been quite useful to solve the Schrödinger equation for the models associated with some other algebras, as those of type A [13], D_4, E_6 and E_7 [14], and now we deal with the E_8 case. The E-series of Lie algebras seems to play a prominent role in fundamental physics through its appearance in several compactification of 11-dimensional SUGRA/M-theory, notably the Horava–Witten interpretation of $E_8 \times E_8$ heterotic superstring theory, and the plausibility of E_6 as the grand unification group for particle physics; see [15] for reviews.

In order to fix the notation, we mention here only a few very basic facts about the exceptional Lie algebra E_8. More details can be found in the monographs; see for instance [16, 17]. The Dynkin diagram of E_8, see figure 1, encodes the Euclidean relations $A_{ij} = (\alpha_i, \alpha_j)$ among the simple roots, which are

$$\begin{aligned} &(\alpha_i, \alpha_i) = 2, && i = 1, \ldots, 8 \\ &(\alpha_i, \alpha_{i+2}) = -1, && i = 1, 2 \\ &(\alpha_i, \alpha_{i+1}) = -1, && i = 3, \ldots, 7 \\ &(\alpha_i, \alpha_j) = 0, && \text{in all other cases.} \end{aligned} \tag{1}$$

The matrix $A = (A_{ij})$ is the Cartan matrix of the algebra, and its inverse A^{-1} gives the eight fundamental weights by $\lambda_i = \sum_{j=1}^{8} A^{-1}_{ji} \alpha_j$, $i = 1, \ldots, 8$. The characters of the basic fundamental representations R_{λ_i} will be denoted by z_i. They are the independent variables in terms of which we want to express the Hamiltonian because of its invariance under the Weyl group W of the algebra E_8.

2. Review of the theory

In this section, we review briefly the general theory of the quantum trigonometric CS model related to a root system $\mathcal{R}$ associated with a simple Lie algebra L of rank r, and later will study explicitly the E_8 case. For CS systems other than trigonometric, see [6] and also [18]. For the other Lie algebras of the E-series, see [14].

The trigonometric Calogero–Sutherland model related to the root system $\mathcal{R}$ of rank r is the quantum system in an Euclidean space $\mathbb{R}^r$ describing a system of particles moving in a circle, defined by the standard Hamiltonian operator

$$H = \frac{1}{2}\sum_{j=1}^{r} p_j^2 + \sum_{\alpha\in\mathcal{R}^+} \kappa_\alpha(\kappa_\alpha - 1)\sin^{-2}(\alpha, q), \tag{2}$$

where $q = (q_j)$ is the Cartesian coordinate system provided by the canonical basis of $\mathbb{R}^r$ and $p_j = -\mathrm{i}\partial_{q_j}$; $\mathcal{R}^+$ is the set of the positive roots of L, and the coupling constants κ_α are such that $\kappa_\alpha = \kappa_\beta$ if $|\alpha| = |\beta|$. We will restrict ourselves to the case of simply laced root systems (as the E-series is), for which the CS model depends only on one coupling constant κ. The explicit form of the potential for all Lie algebras can be found in [6] (second reference).

To find the stationary states, it is necessary to solve the Schrödinger eigenvalue problem $H\Psi = E\Psi$. The following important facts about this family of quantum mechanical systems were established in [6].

(a) The ground state energy and (non-normalized) wavefunction of these integrable systems are

$$\begin{aligned} E_0(\kappa) &= 2\rho^2\kappa^2 \\ \Psi_0^\kappa(q) &= \prod_{\alpha\in\mathcal{R}^+} \sin^\kappa(\alpha, q), \end{aligned} \tag{3}$$

with ρ being the Weyl vector $\rho = (1/2)\sum_{\alpha\in\mathcal{R}^+}\alpha$ of the algebra, while the excited states are indexed by the highest weights, $\mu = \sum m_i\lambda_i \in P^+$ (where P^+ is the cone of dominant weights), of the irreducible representations of L, that is, by the r-tuple of non-negative integers $\mathbf{m} = (m_1, \ldots, m_r)$ (the quantum numbers), and the wavefunctions $\Psi_{\mathbf{m}}^\kappa$ and the energy levels $E_{\mathbf{m}}(\kappa)$ satisfy

$$\begin{aligned} H\Psi_{\mathbf{m}}^\kappa &= E_{\mathbf{m}}(\kappa)\Psi_{\mathbf{m}}^\kappa \\ E_{\mathbf{m}}(\kappa) &= 2(\mu + \kappa\rho, \mu + \kappa\rho). \end{aligned} \tag{4}$$

(b) It is natural to look for the solutions $\Psi_{\mathbf{m}}^\kappa$ in the form

$$\Psi_{\mathbf{m}}^\kappa(q) = \Psi_0^\kappa(q)\Phi_{\mathbf{m}}^\kappa(q), \tag{5}$$

and consequently we are led to the eigenvalue problem

$$\Delta^\kappa\Phi_{\mathbf{m}}^\kappa = \varepsilon_{\mathbf{m}}(\kappa)\Phi_{\mathbf{m}}^\kappa, \tag{6}$$

where Δ^κ is the linear differential operator

$$\Delta^\kappa = -\frac{1}{2}\sum_{j=1}^{r}\partial_{q_j}^2 - \kappa\sum_{\alpha\in\mathcal{R}^+}\cot(\alpha, q)(\alpha, \partial_q), \tag{7}$$

and the eigenvalues $\varepsilon_{\mathbf{m}}(\kappa)$ are the energies over the ground level, i.e.,

$$\varepsilon_{\mathbf{m}}(\kappa) = E_{\mathbf{m}}(\kappa) - E_0(\kappa) = 2(\mu, \mu + 2\kappa\rho). \tag{8}$$

Taking into account that $(\lambda_j, \lambda_k) = A_{jk}^{-1}$, it is possible to give a more explicit expression for the eigenvalues $\varepsilon_{\mathbf{m}}(\kappa)$:

$$\varepsilon_{\mathbf{m}}(\kappa) = 2\sum_{j,k=1}^{r} A_{jk}^{-1} m_j m_k + 4\kappa\sum_{j,k=1}^{r} A_{jk}^{-1} m_j. \tag{9}$$

We will write $\varepsilon_j(\kappa)$ for the fundamental weight λ_j, i.e., $\varepsilon_j(\kappa) = 2\big(A_{jj}^{-1} + 2\kappa\sum_k A_{jk}^{-1}\big)$ for the quantum numbers $(0, \ldots, \overset{(j)}{1}, \ldots, 0)$.

(c) In the case $\kappa = 0$, the wavefunctions (6) are (proportional to) the monomial symmetric functions:

$$M_\lambda(q) = \sum_{w \in W} \mathrm{e}^{2\mathrm{i}(w\cdot\lambda, q)}, \qquad \lambda \in P^+, \tag{10}$$

W being the Weyl group of L. And the wavefunctions in the case $\kappa = 1$ are (proportional to) the characters of the irreducible representations:

$$\chi_\lambda(q) = \frac{\sum_{w\in W} (\det w)\, \mathrm{e}^{2\mathrm{i}(w\cdot(\lambda+\rho), q)}}{\sum_{w\in W} (\det w)\, \mathrm{e}^{2\mathrm{i}(w\cdot\rho, q)}}, \qquad \lambda \in P^+. \tag{11}$$

Both M_λ and χ_λ are sums over the orbit $\{w \cdot \lambda\}$ of λ under W, and consequently, W-invariant; as wavefunctions, they represent superpositions of plane waves.

Due to the Weyl symmetry of the Hamiltonian, the wavefunctions $\Phi^\kappa_{\mathbf{m}}(q)$ are W-invariant, and the best way to solve the eigenvalue problem (6) is to use the set of independent W-invariant variables, $z_k = \chi_{\lambda_k}(q)$, in terms of which the wavefunctions $\Phi^\kappa_{\mathbf{m}}$ are polynomials. Unfortunately, the expression of these characters z_k in terms of q-variables is complicated and makes the direct change of variables $z = z(q)$ very cumbersome. We are thus forced to follow a much more convenient, indirect route, which has proven to be very useful for other root systems [14].

First of all, we need the expression of the operator Δ^κ in terms of z-variables. To this goal, the starting point is to write Δ^κ (7) in its general form:

$$\Delta^\kappa = \sum_{j\leqslant k} \boldsymbol{a}_{jk}(z)\partial_{z_j}\partial_{z_k} + \sum_j \left[b^1_j(z) + (\kappa - 1)b_j(z)\right]\partial_{z_j}. \tag{12}$$

Now, if we take into account the fact that, as pointed above, $b^1_j(z) = \Delta^1 z_j = \varepsilon_j(1)z_j$, the full expression for the coefficients $b^1_j(z)$ appearing in Δ^1 is completely determined by the inverse Cartan matrix of the algebra; explicitly,

$$b^1_j(z) = 2\left(A^{-1}_{jj} + 2\sum_{k\neq j} A^{-1}_{kj}\right) z_j, \qquad j = 1, \ldots, r. \tag{13}$$

On the other hand, in order to find the coefficients $\boldsymbol{a}_{jk}$ we note that $\Delta^1(z_j z_k) = \boldsymbol{a}_{jk}(z) + b^1_j(z)z_k + b^1_k(z)z_j$. As the product character $z_j z_k$ is the character of the tensor product $R_{\lambda_j} \otimes R_{\lambda_k}$ of representations, knowing the full set of the quadratic Clebsch–Gordan series we will be able to determine the $\boldsymbol{a}$-coefficients. The Clebsch–Gordan series yields the formulae,

$$z_j z_k = \sum_{\mu\in Q_{jk}} N_{\mu;jk}\chi_\mu(z), \tag{14}$$

for the products of fundamental characters $z_j z_k$, with $Q_{jk} \subset P^+$ being the set of dominant weights in the irreducible representation of highest weight $\lambda_j + \lambda_k$, and $N_{\lambda;jk}$ the multiplicity of the irreducible representation R_λ in that series; in particular, $N_{\lambda_j+\lambda_k;jk} = 1$. Consequently, we obtain the coefficients $\boldsymbol{a}$ by applying Δ^1 to the two members of (14). The required Clebsch–Gordan series will be obtained using LiE [19], a computer algebra system 'specialized in computations involving Lie groups and their representations'.

The remaining step is to look for the coefficients $b_j(z)$. These can be found if we know enough monomial symmetric functions M_λ in terms of the z-variables. Suppose that the functions $M_{\lambda_k} = M_k(z), k = 1, \ldots, r$, are known, then from the eigenvalue equation $\Delta^0 M_k = \varepsilon_k(0) M_k$ we obtain the following linear system for the coefficients $b_j^0(z) = b_j^1(z) - b_j(z)$:

$$\sum_{i \leqslant j} \boldsymbol{a}_{ij}(z) \frac{\partial^2 M_k}{\partial z_i \partial z_j} + \sum b_j^0(z) \frac{\partial M_k}{\partial z_j} = 2\lambda_k^2 M_k(z). \tag{15}$$

This system has a unique solution $\left(b_j^0\right)$ because each of the sets of characters and monomial symmetric functions constitutes a basis of W-invariant functions. These functions $M_k(z)$ can be found following the strategy that we have developed in the E_6 and E_7 cases [14]. But it is necessary to know the multiplicities in the expansion of each z in terms of the monomial functions M_{λ_k}; to obtain these multiplicities we rely on the system LiE [19] again.

Summing up, to find an explicit expression (12) of the Hamiltonian Δ^κ for general κ, we have to deal only with the cases $\kappa = 0$ and $\kappa = 1$, plus the additional information about the underlying algebra L provided by the quadratic Clebsch–Gordan series and some monomial symmetric functions. This is done in the following section for the algebra E_8.

3. The Calogero–Sutherland Hamiltonian in E_8

The coefficients $b_j(z)$ in the expression of Δ^1 are easily obtained from (13) (for the inverse Cartan matrix A^{-1}, see, for instance, [16, 17] or [19]):

$$b_1^1(z) = 192 z_1, \qquad b_2^1(z) = 288 z_2, \qquad b_3^1(z) = 392 z_3, \qquad b_4^1(z) = 600 z_4,$$
$$b_5^1(z) = 480 z_5, \qquad b_6^1(z) = 360 z_6, \qquad b_7^1(z) = 240 z_7, \qquad b_8^1(z) = 120 z_8.$$

We can now follow the lines indicated above to obtain the $\boldsymbol{a}$-coefficients. Recall that we need only to look for the operator $\Delta^1 = \sum_{i \leqslant j} \boldsymbol{a}_{ij} \partial_{z_i} \partial_{z_j} + \sum_j b_j^1 \partial_{z_j}$. The best way to obtain the $\boldsymbol{a}$'s is to start from the outer regions of the Dynkin diagram (figure 1) in such a way that the computation of each $\boldsymbol{a}$ only requires the knowledge of those already obtained, as explained in [14]. The ordered list of the 28 independent coefficients $\boldsymbol{a}_{jk}(z)$ in Δ^1 is

$$\boldsymbol{a}_{88}(z) = -4\left(31 + 7z_1 + z_7 + 16z_8 - z_8^2\right),$$
$$\boldsymbol{a}_{18}(z) = 8\left(-16z_1 - 4z_2 - 10z_7 - 25z_8 + z_1 z_8\right),$$
$$\boldsymbol{a}_{11}(z) = 4\left(-31 + 3z_1 + 2z_1^2 - 4z_2 - z_3 - 5z_6 + 9z_7 - 6z_8 - 10z_1 z_8 - 19z_8^2\right),$$
$$\boldsymbol{a}_{78}(z) = -4\left(-31 + z_1 + 8z_2 + 3z_6 - 11z_7 + 19z_8 + 13z_1 z_8 - 3z_7 z_8 + 31z_8^2\right),$$
$$\boldsymbol{a}_{28}(z) = -4\left(32z_1 + 8z_2 + 7z_3 + 15z_6 + 20z_7 - 25z_8 + 25z_1 z_8 - 3z_2 z_8\right),$$
$$\begin{aligned}\boldsymbol{a}_{17}(z) = 4\big(&31 + 31z_1 + 24z_2 - 7z_3 + 4z_6 - 9z_7 + 4z_1 z_7 + 6z_8 - 38z_1 z_8 - 7z_2 z_8 \\ &- 19z_7 z_8 - 31z_8^2\big),\end{aligned}$$
$$\begin{aligned}\boldsymbol{a}_{68}(z) = -4\big(&31 - z_1 + 20z_2 + 7z_3 + 4z_5 - 4z_6 + 51z_7 + 12z_1 z_7 + 6z_8 - 6z_1 z_8 + 7z_2 z_8 \\ &- 4z_6 z_8 + 19z_7 z_8 - 31z_8^2\big),\end{aligned}$$
$$\begin{aligned}\boldsymbol{a}_{77}(z) = -4\big(&31 + 6z_1 + 7z_1^2 + 6z_2 - 14z_3 + 2z_5 - 3z_6 - 22z_7 - 3z_7^2 - 34z_8 - 12z_1 z_8 \\ &+ 8z_2 z_8 + z_6 z_8 - 10z_7 z_8 + 4z_8^2 + 6z_1 z_8^2 + 15z_8^3\big),\end{aligned}$$
$$\begin{aligned}\boldsymbol{a}_{12}(z) = -4\big(&-31 - 6z_1 + 25z_1^2 - 24z_2 - 5z_1 z_2 - 18z_3 + 5z_5 - 29z_6 - 31z_7 + 13z_1 z_7 \\ &+ 19z_8 + 13z_1 z_8 + 7z_2 z_8 + 19z_7 z_8 + 6z_8^2\big),\end{aligned}$$
$$\begin{aligned}\boldsymbol{a}_{38}(z) = -4\big(&-7z_1 + 25z_1^2 + 28z_2 + 7z_1 z_2 - 18z_3 + 5z_5 + 10z_6 - 20z_7 + 13z_1 z_7 \\ &+ 7z_2 z_8 - 4z_3 z_8 - 20z_7 z_8 - 25z_8^2\big),\end{aligned}$$

$$\begin{aligned}
\boldsymbol{a}_{27}(z) &= -4\big(-31z_1 + z_1^2 - 10z_2 + 7z_1z_2 + 13z_3 - 9z_5 - 15z_6 + 20z_7 + z_1z_7 - 6z_2z_7 \\
&\quad + 25z_8 + 7z_1z_8 - 10z_2z_8 + 6z_3z_8 + 14z_6z_8 + 20z_7z_8 - 25z_8^2 + 24z_1z_8^2\big), \\
\boldsymbol{a}_{16}(z) &= -4\big(33z_1 + z_1^2 - 18z_2 + 7z_1z_2 - 15z_3 - 9z_5 - 15z_6 - 6z_1z_6 + 18z_7 + 19z_1z_7 + 6z_2z_7 \\
&\quad + 18z_7^2 - 7z_8 - 21z_1z_8 - 10z_2z_8 + 6z_3z_8 - 4z_6z_8 + 56z_7z_8 - 7z_8^2 + 6z_1z_8^2 - 18z_8^3\big), \\
\boldsymbol{a}_{22}(z) &= -4\big(31 - 4z_1 + 13z_1^2 + 3z_2 + 8z_1z_2 - 4z_2^2 - 2z_3 + z_4 - 9z_5 + 19z_6 + 4z_1z_6 + 16z_1z_7 \\
&\quad + 3z_2z_7 + 9z_7^2 - 30z_8 - 15z_1z_8 + 9z_1^2z_8 - 16z_2z_8 - 6z_3z_8 - 11z_6z_8 - 30z_7z_8 \\
&\quad + 10z_8^2 - 6z_1z_8^2 + 10z_8^3\big), \\
\boldsymbol{a}_{13}(z) &= -4\big(31 + 36z_1 - 27z_1^2 - 18z_2 + 16z_1z_2 + 22z_3 - 7z_1z_3 + 3z_4 - 18z_5 - 9z_6 + 9z_1z_6 \\
&\quad + 22z_7 - 36z_1z_7 + 6z_2z_7 - 9z_7^2 - 12z_8 + 13z_1z_8 + 19z_1^2z_8 + 9z_2z_8 - 13z_3z_8 + 4z_6z_8 \\
&\quad - 20z_7z_8 - 18z_8^2 + 14z_1z_8^2 - 10z_8^3\big), \\
\boldsymbol{a}_{58}(z) &= -4\big(-31 - 12z_1 - 13z_1^2 + 6z_2 + 7z_1z_2 + 20z_3 + 5z_4 - 9z_5 + 65z_6 + 11z_1z_6 - 42z_7 \\
&\quad - 12z_1z_7 + 6z_2z_7 - 11z_7^2 - 22z_8 + 24z_2z_8 + 6z_3z_8 - 5z_5z_8 + 25z_6z_8 - 42z_7z_8 \\
&\quad + 50z_8^2 - 6z_1z_8^2 + 11z_8^3\big), \\
\boldsymbol{a}_{67}(z) &= -4\big(-8z_1 - 8z_1^2 - 8z_2 + z_1z_2 + 2z_3 + 5z_4 - 9z_5 + 30z_6 - 49z_7 + 7z_1z_7 + 8z_2z_7 \\
&\quad - 8z_6z_7 - 9z_7^2 + 22z_8 + 33z_1z_8 + 13z_1^2z_8 + 24z_2z_8 - 19z_3z_8 + 3z_5z_8 - 9z_6z_8 \\
&\quad - 40z_7z_8 + 11z_1z_7z_8 + 7z_8^2 - 31z_1z_8^2 + 7z_2z_8^2 + 19z_7z_8^2 - 2z_8^3\big), \\
\boldsymbol{a}_{37}(z) &= -4\big(-31 - 5z_1 - 6z_1^2 + 2z_2 - 24z_1z_2 + 8z_2^2 - 19z_3 + 7z_1z_3 - 16z_4 + 26z_5 - 30z_6 \\
&\quad - 7z_1z_6 - 11z_7 + 14z_1z_7 - 14z_2z_7 - 8z_3z_7 + 20z_7^2 + 31z_8 - 21z_1z_8 + 18z_1^2z_8 \\
&\quad + 7z_2z_8 + 6z_1z_2z_8 + 2z_3z_8 + 5z_5z_8 - 11z_6z_8 + 66z_7z_8 + 13z_1z_7z_8 + 36z_8^2 + z_1z_8^2 \\
&\quad + 7z_2z_8^2 - 19z_7z_8^2 - 46z_8^3\big), \\
\boldsymbol{a}_{26}(z) &= 4\big(31 - z_1 - 31z_2 - 8z_2^2 - 20z_3 - 7z_1z_3 + 16z_4 - 31z_5 + 25z_6 + 18z_1z_6 + 9z_2z_6 \\
&\quad - 11z_7 - 34z_1z_7 - 19z_2z_7 - 5z_3z_7 - 13z_6z_7 - 22z_7^2 - 57z_8 + 7z_1z_8 + 6z_1^2z_8 \\
&\quad + 22z_2z_8 - 6z_1z_2z_8 - 6z_3z_8 + 8z_5z_8 + 24z_6z_8 - 52z_7z_8 - 24z_1z_7z_8 - 12z_8^2 \\
&\quad + 13z_1z_8^2 + 17z_2z_8^2 + 22z_8^3\big), \\
\boldsymbol{a}_{15}(z) &= -4\big(-19z_1 + 13z_1^2 + z_2 + 6z_1z_2 + 8z_2^2 - 15z_3 + 7z_1z_3 - 16z_4 - 21z_5 - 8z_1z_5 \\
&\quad - 43z_6 + 11z_1z_6 + 5z_2z_6 + 20z_7 - 19z_1z_7 - 19z_2z_7 + 5z_3z_7 + 13z_6z_7 + 20z_7^2 + 32z_8 \\
&\quad + z_1z_8 - 19z_1^2z_8 - 5z_2z_8 + 6z_1z_2z_8 + 13z_3z_8 - 8z_5z_8 + 44z_6z_8 + 32z_7z_8 - 5z_1z_7z_8 \\
&\quad - 25z_8^2 + 12z_1z_8^2 + 12z_2z_8^2 - 19z_7z_8^2 + 9z_8^3\big), \\
\boldsymbol{a}_{23}(z) &= -4\big(-36z_1 - 12z_1^2 + 24z_1^3 - 13z_2 - 3z_1z_2 - 5z_3 - 34z_1z_3 - 10z_2z_3 + 19z_4 - 6z_5 \\
&\quad + 4z_1z_5 + 51z_6 - 17z_1z_6 + 5z_2z_6 - 29z_7 - 37z_1z_7 + 12z_1^2z_7 + 11z_2z_7 - 7z_3z_7 + z_6z_7 \\
&\quad - 29z_7^2 - 20z_8 + 43z_1z_8 - 11z_1^2z_8 + 5z_2z_8 + 14z_1z_2z_8 - z_3z_8 - 16z_5z_8 + 12z_6z_8 \\
&\quad - 60z_7z_8 - 4z_1z_7z_8 - 7z_8^2 - 13z_1z_8^2 - 9z_2z_8^2 + 26z_8^3\big), \\
\boldsymbol{a}_{48}(z) &= 4\big(-31 + 7z_1 + 12z_1^2 + 6z_1^3 + 25z_2 - 6z_1z_2 - 8z_2^2 - 24z_3 - 19z_1z_3 - 6z_2z_3 + 22z_4 \\
&\quad - 37z_5 - 4z_1z_5 + 7z_6 + 14z_1z_6 - 5z_2z_6 - 13z_7 + 6z_1^2z_7 - 15z_2z_7 - 11z_3z_7 + 9z_6z_7 \\
&\quad - 2z_7^2 + 15z_8 - 17z_1z_8 + 26z_1^2z_8 + 18z_2z_8 - 6z_1z_2z_8 - 44z_3z_8 + 6z_4z_8 - 20z_5z_8 \\
&\quad + 16z_6z_8 - 32z_7z_8 + 10z_1z_7z_8 + 40z_8^2 - 21z_1z_8^2 - 5z_2z_8^2 - 20z_7z_8^2 - 12z_8^3\big), \\
\boldsymbol{a}_{57}(z) &= 4\big(-31 - 11z_1 - 6z_1^2 + 6z_1^3 + 13z_1z_2 + 8z_2^2 + 18z_3 - 12z_1z_3 - 6z_2z_3 - 5z_4 - 31z_5
\end{aligned}$$

$$+ 6z_1z_5 + 21z_6 - 10z_1z_6 - 8z_2z_6 - 2z_7 - 31z_1z_7 - 6z_1^2z_7 - 20z_2z_7 + 18z_3z_7$$
$$+ 10z_5z_7 + 6z_6z_7 + 29z_7^2 + 41z_8 + 18z_1z_8 + z_1^2z_8 - 8z_1z_2z_8 - z_3z_8 - 4z_4z_8 + 17z_5z_8$$
$$- 23z_6z_8 - 10z_1z_6z_8 + 51z_7z_8 + 25z_1z_7z_8 - 6z_2z_7z_8 + 10z_7^2z_8 + 32z_8^2 - 7z_1z_8^2$$
$$- 17z_2z_8^2 - 6z_3z_8^2 - 24z_6z_8^2 + 21z_7z_8^2 - 36z_8^3 + 6z_1z_8^3 - 10z_8^4\big),$$

$$\begin{aligned}
\boldsymbol{a}_{66}(z) = -4\big(&31 - 8z_1 - 3z_1^2 + 4z_1^3 - z_2 - 3z_1z_2 + 7z_3 - 15z_1z_3 + 3z_2z_3 + 18z_4 - 17z_5\\
&- 3z_1z_5 + 41z_6 + 3z_1z_6 - 6z_6^2 + 19z_7 - 6z_1z_7 - 3z_1^2z_7 + 17z_2z_7 + 10z_3z_7 + z_5z_7\\
&- z_6z_7 + 18z_7^2 + 5z_1z_7^2 - 45z_8 - 5z_1^2z_8 + 9z_2z_8 + z_1z_2z_8 + 5z_3z_8 + 2z_4z_8 - 9z_5z_8\\
&+ 35z_6z_8 - 62z_7z_8 + 3z_1z_7z_8 + 7z_2z_7z_8 + 5z_7^2z_8 - 12z_8^2 + 14z_1z_8^2 + 6z_1^2z_8^2 - 2z_2z_8^2\\
&- 12z_3z_8^2 - 2z_6z_8^2 - 59z_7z_8^2 + 21z_8^3 - 12z_1z_8^3 + 14z_8^4\big),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{a}_{36}(z) = 4\big(&31 - 27z_1 - 51z_1^2 + 7z_1^3 - 42z_2 + 43z_1z_2 - 7z_1^2z_2 - 16z_2^2 + 42z_3 - 8z_1z_3 - z_2z_3\\
&+ 32z_4 - 43z_5 + 17z_1z_5 + 79z_6 + 25z_1z_6 - 9z_2z_6 + 12z_3z_6 + 2z_7 - 40z_1z_7 - 18z_1^2z_7\\
&- 18z_2z_7 - 5z_1z_2z_7 + 29z_3z_7 - 5z_5z_7 + 11z_7^2 - 13z_1z_7^2 - 57z_8 + 64z_1z_8 - 7z_1^2z_8\\
&- 22z_2z_8 + 24z_1z_2z_8 - 7z_2^2z_8 + 4z_3z_8 - 6z_1z_3z_8 + 14z_4z_8 - 48z_5z_8 + 42z_6z_8\\
&+ 6z_1z_6z_8 - 55z_7z_8 - 19z_1z_7z_8 + 6z_2z_7z_8 - z_7^2z_8 - 14z_8^2 + 20z_1z_8^2 + 6z_1^2z_8^2\\
&+ 18z_2z_8^2 - 18z_3z_8^2 + 20z_6z_8^2 - 58z_7z_8^2 + 20z_8^4\big),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{a}_{25}(z) = 4\big(&31 + 10z_1 + 18z_1^2 + 7z_1^3 + 2z_2 - 5z_1z_2 - 7z_1^2z_2 + 12z_2^2 - 31z_3 - z_1z_3 - z_2z_3\\
&- 30z_4 - 23z_5 + 17z_1z_5 + 12z_2z_5 - 54z_6 - 73z_1z_6 - 9z_2z_6 - 4z_3z_6 - 12z_6^2 + 42z_7\\
&+ 29z_1z_7 + 18z_1^2z_7 + 2z_2z_7 - 5z_1z_2z_7 - 31z_3z_7 + 7z_5z_7 - 24z_6z_7 + 11z_7^2 - z_1z_7^2\\
&+ 54z_8 + 20z_1z_8 - 20z_1^2z_8 - 7z_2z_8 - 4z_1z_2z_8 - 7z_2^2z_8 + 13z_3z_8 - 6z_1z_3z_8 + 14z_4z_8\\
&- 7z_5z_8 + 69z_6z_8 - 6z_1z_6z_8 + 33z_7z_8 + 27z_1z_7z_8 + 6z_2z_7z_8 - z_7^2z_8 - 77z_8^2 - 18z_1z_8^2\\
&+ 6z_1^2z_8^2 + 4z_2z_8^2 + 6z_3z_8^2 + 8z_6z_8^2 - 72z_7z_8^2 + z_8^3 - 12z_1z_8^3 + 20z_8^4\big),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{a}_{14}(z) = -4\big(&7z_1 - 46z_1^2 - 21z_1^3 - 9z_2 - 13z_1z_2 + 7z_1^2z_2 - 12z_2^2 + 55z_3 + 36z_1z_3 + z_2z_3 - z_4\\
&- 10z_1z_4 - 26z_5 + 11z_1z_5 + 4z_2z_5 - 7z_6 + 6z_1z_6 - 20z_2z_6 + 4z_3z_6 - 4z_6^2 + 22z_7\\
&- 8z_1z_7 - 18z_1^2z_7 - 33z_2z_7 + 5z_1z_2z_7 + 9z_5z_7 + 19z_6z_7 + 22z_7^2 - 9z_1z_7^2 + 28z_8\\
&- 32z_1z_8 + 20z_1^2z_8 - 44z_2z_8 + 16z_1z_2z_8 + 7z_2^2z_8 + 41z_3z_8 + 6z_1z_3z_8 - 14z_4z_8\\
&+ 4z_5z_8 + 2z_6z_8 - 12z_1z_6z_8 + 54z_7z_8 + 19z_1z_7z_8 + 14z_2z_7z_8 + 9z_7^2z_8 + 5z_8^2 + 39z_1z_8^2\\
&- 18z_1^2z_8^2 + z_2z_8^2 + 24z_3z_8^2 - 8z_6z_8^2 + 16z_7z_8^2 - 26z_8^3 + 2z_1z_8^3 + 10z_8^4\big),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{a}_{33}(z) = -4\big(&31 - z_1 - 17z_1^2 - 17z_1^3 - 28z_2 - 6z_1z_2 + 12z_1^2z_2 + 2z_2^2 + 42z_3 + 14z_1z_3 - 8z_2z_3\\
&- 7z_3^2 + 23z_4 + z_1z_4 - 40z_5 - 14z_1z_5 + 2z_2z_5 + 41z_6 - 7z_1z_6 + 4z_1^2z_6 + 5z_2z_6 - 2z_3z_6\\
&+ 3z_6^2 + 11z_7 - 5z_1z_7 - 30z_1^2z_7 - 22z_2z_7 + 6z_1z_2z_7 + 21z_3z_7 - 8z_5z_7 - 7z_6z_7 + 20z_7^2\\
&- 7z_1z_7^2 - 61z_8 + 26z_1z_8 + 14z_1^2z_8 + 9z_1^3z_8 - 11z_2z_8 - 3z_1z_2z_8 - 12z_1z_3z_8\\
&+ 7z_4z_8 - 13z_5z_8 + 39z_6z_8 + 3z_1z_6z_8 - 28z_7z_8 + 18z_1z_7z_8 - 6z_2z_7z_8 - z_7^2z_8 - 4z_8^2\\
&+ 9z_1z_8^2 + 5z_1^2z_8^2 + 3z_2z_8^2 - 8z_3z_8^2 + 9z_6z_8^2 - 50z_7z_8^2 + 6z_8^3 - 16z_1z_8^3 + 20z_8^4\big),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{a}_{47}(z) = -4\big(&15z_1 + 36z_1^2 + 21z_1^3 + 25z_2 - 11z_1z_2 - 7z_1^2z_2 - 16z_2^2 + 7z_1z_2^2 - 62z_3 - 49z_1z_3\\
&+ 7z_3^2 + 31z_4 - 14z_1z_4 + 3z_5 + 22z_1z_5 + z_2z_5 + 17z_6 - z_1z_6 - 3z_1^2z_6 + 3z_2z_6 - 8z_3z_6\\
&+ 4z_6^2 - 11z_7 - 9z_1z_7 + 8z_1^2z_7 + 29z_2z_7 - 6z_1z_2z_7 - 12z_4z_7 - 7z_5z_7 - 13z_6z_7\\
&- 51z_7^2 + 11z_1z_7^2 + 25z_8 + 2z_1z_8 + 2z_1^2z_8 - 19z_1^3z_8 + 43z_2z_8 + z_1z_2z_8 + z_2^2z_8 - 33z_3z_8\\
&+ 45z_1z_3z_8 + 5z_2z_3z_8 - 25z_4z_8 + 28z_5z_8 + 4z_1z_5z_8 - 35z_6z_8 - 22z_1z_6z_8 + 5z_2z_6z_8
\end{aligned}$$

$$- 108z_7z_8 + 51z_1z_7z_8 - 5z_1^2z_7z_8 + 16z_2z_7z_8 + 10z_3z_7z_8 - 9z_6z_7z_8 + 5z_7^2z_8 - 5z_8^2$$
$$- 39z_1z_8^2 - 8z_1^2z_8^2 - 32z_2z_8^2 + 6z_1z_2z_8^2 + 33z_3z_8^2 + 19z_5z_8^2 - 18z_6z_8^2 + 71z_7z_8^2$$
$$- 11z_1z_7z_8^2 + 2z_8^3 + 9z_1z_8^3 + 5z_2z_8^3 + 19z_7z_8^3 - 8z_8^4\big),$$

$$\boldsymbol{a}_{56}(z) = 4\big(14z_1 + 7z_1^2 - 7z_1^3 + 6z_2 - 15z_1z_2 + 6z_1^2z_2 - 7z_1z_2^2 - 8z_3 + z_1z_3 - 5z_2z_3 - 7z_3^2$$
$$+ 8z_4 + 14z_1z_4 + 21z_5 - 6z_1z_5 + 7z_2z_5 - 33z_6 + 50z_1z_6 + 14z_1^2z_6 - 25z_2z_6 - 21z_3z_6$$
$$+ 15z_5z_6 - 3z_6^2 + 29z_7 - 7z_1z_7 + 7z_1^2z_7 - z_1z_2z_7 - 8z_3z_7 - 3z_4z_7 + 11z_5z_7 - 64z_6z_7$$
$$- 9z_1z_6z_7 + 38z_7^2 - z_1z_7^2 - 6z_2z_7^2 + 9z_7^3 - 29z_8 - 59z_1z_8 + 14z_1^2z_8 + 6z_1^3z_8 + 2z_2z_8$$
$$+ 19z_1z_2z_8 + 8z_2^2z_8 + 5z_3z_8 + z_1z_3z_8 - 5z_2z_3z_8 - 31z_4z_8 - 26z_5z_8 + 5z_1z_5z_8$$
$$- 74z_6z_8 - 16z_1z_6z_8 - 7z_2z_6z_8 + 70z_7z_8 - 72z_1z_7z_8 - 6z_1^2z_7z_8 - 30z_2z_7z_8$$
$$+ 11z_3z_7z_8 - 17z_6z_7z_8 + 79z_7^2z_8 + 61z_8^2 + 18z_1z_8^2 - 18z_1^2z_8^2 - 24z_2z_8^2 - z_1z_2z_8^2 + 6z_3z_8^2$$
$$+ 8z_5z_8^2 + 21z_6z_8^2 + 49z_7z_8^2 + 19z_1z_7z_8^2 - 17z_8^3 + 19z_1z_8^3 + 7z_2z_8^3 - 28z_7z_8^3 - 18z_8^4\big),$$

$$\boldsymbol{a}_{35}(z) = 4\big(-25z_1 - 6z_1^2 - 13z_1^3 - z_2 + 13z_1z_2 + 8z_1^2z_2 - 6z_2^2 - 8z_1z_2^2 + 21z_3 + 15z_1z_3$$
$$- 7z_1^2z_3 - 21z_2z_3 + 7z_3^2 + 26z_4 + 7z_1z_4 + 7z_5 + 13z_1z_5 + 10z_2z_5 + 16z_3z_5 + 46z_6$$
$$+ 41z_1z_6 - 7z_1^2z_6 - 26z_2z_6 - 4z_1z_2z_6 + 7z_3z_6 - 5z_5z_6 - 25z_6^2 - 18z_7 - 36z_1z_7$$
$$- 6z_1^2z_7 + 11z_2z_7 + 19z_1z_2z_7 - 6z_2^2z_7 + 21z_3z_7 - 5z_1z_3z_7 + 12z_4z_7 - 2z_5z_7$$
$$+ 52z_6z_7 - 8z_1z_6z_7 - 36z_7^2 - 11z_1z_7^2 + 12z_2z_7^2 - 18z_7^3 - 27z_8 - 9z_1z_8 + 5z_1^2z_8$$
$$+ 13z_1^3z_8 - 2z_2z_8 + 9z_1z_2z_8 - 6z_1^2z_2z_8 - 5z_2^2z_8 - 20z_3z_8 - z_1z_3z_8 - z_2z_3z_8$$
$$+ 2z_4z_8 - 21z_5z_8 + 15z_1z_5z_8 + 26z_6z_8 - 13z_1z_6z_8 - 15z_2z_6z_8 - 91z_7z_8 - 3z_1z_7z_8$$
$$+ 11z_1^2z_7z_8 + 24z_2z_7z_8 - 6z_3z_7z_8 + 45z_6z_7z_8 - 64z_7^2z_8 - 16z_8^2 + 45z_1z_8^2 - 19z_1^2z_8^2$$
$$- 17z_2z_8^2 + z_1z_2z_8^2 + 7z_3z_8^2 - 22z_5z_8^2 + 17z_6z_8^2 - 10z_7z_8^2 - 5z_1z_7z_8^2 + 14z_8^3 - 5z_1z_8^3$$
$$- z_2z_8^3 + 18z_7z_8^3 + 14z_8^4\big),$$

$$\boldsymbol{a}_{24}(z) = -4\big(32z_1 - 6z_1^2 - 6z_1^3 + 25z_2 - 46z_1z_2 + 12z_1^2z_2 - 2z_2^2 + 8z_1z_2^2 - 17z_3 + 3z_1z_3$$
$$+ 7z_1^2z_3 - 14z_2z_3 - 7z_3^2 + 4z_4 - 7z_1z_4 - 15z_2z_4 + 24z_5 + 5z_1z_5 - 2z_2z_5 + 3z_3z_5$$
$$- 3z_6 + 13z_1z_6 - 17z_1^2z_6 + 31z_2z_6 + 4z_1z_2z_6 + 14z_3z_6 + 5z_5z_6 + 7z_6^2 - 13z_7 - 7z_1z_7$$
$$- 14z_2z_7 + 9z_1z_2z_7 + 6z_2^2z_7 - 2z_3z_7 + 5z_1z_3z_7 - 12z_4z_7 + 25z_5z_7 - 21z_6z_7$$
$$- 13z_1z_6z_7 + 7z_7^2 - 6z_1z_7^2 + 9z_2z_7^2 - 39z_8 - 26z_1z_8 + 23z_1^2z_8 - 26z_1^3z_8 - 17z_2z_8$$
$$- z_1z_2z_8 + 6z_1^2z_2z_8 - 21z_2^2z_8 + 7z_3z_8 + 26z_1z_3z_8 + z_2z_3z_8 + 28z_4z_8 - 6z_5z_8$$
$$+ 6z_1z_5z_8 + 7z_6z_8 + 49z_1z_6z_8 - 14z_2z_6z_8 + 29z_7z_8 - 8z_1z_7z_8 - 22z_1^2z_7z_8 - 31z_2z_7z_8$$
$$+ 12z_3z_7z_8 + 10z_6z_7z_8 + 23z_7^2z_8 + 6z_8^2 - 52z_1z_8^2 + 32z_1^2z_8^2 + 2z_2z_8^2 + 4z_1z_2z_8^2$$
$$- 10z_3z_8^2 - 20z_5z_8^2 - 4z_6z_8^2 - 52z_7z_8^2 + 10z_1z_7z_8^2 + 3z_8^3 + 15z_1z_8^3 + 7z_2z_8^3 + 10z_8^4\big),$$

$$\boldsymbol{a}_{46}(z) = -4\big(-31z_1 + 7z_1^2 - 2z_1^3 - 8z_1^4 - 35z_2 + 11z_1z_2 - 6z_1^2z_2 + 20z_2^2 - z_1z_2^2 + 8z_2^3 + 8z_3$$
$$+ 19z_1z_3 + 24z_1^2z_3 + 14z_2z_3 + 8z_1z_2z_3 - 8z_3^2 - 39z_4 - 22z_1z_4 - 24z_2z_4 + 12z_5$$
$$+ 8z_1z_5 - 8z_1^2z_5 + 37z_2z_5 + 15z_3z_5 - 17z_6 - 70z_1z_6 - 29z_2z_6 - 15z_1z_2z_6 - 18z_3z_6$$
$$- 18z_4z_6 + 3z_5z_6 + 15z_6^2 + 66z_7 - 5z_1z_7 - 10z_1^2z_7 - 34z_2z_7 + 16z_1z_2z_7 + 8z_2^2z_7$$
$$+ 31z_3z_7 - 5z_1z_3z_7 + 4z_2z_3z_7 - z_5z_7 + 4z_1z_5z_7 + 41z_6z_7 - 2z_1z_6z_7 + 5z_2z_6z_7$$
$$+ 6z_7^2 - 4z_1^2z_7^2 + 17z_2z_7^2 + 9z_3z_7^2 - 9z_6z_7^2 + 47z_8 + 81z_1z_8 - 14z_1^2z_8 + 28z_1^3z_8$$
$$+ 43z_2z_8 - 6z_1z_2z_8 - 21z_1^2z_2z_8 - 23z_2^2z_8 + 6z_1z_2^2z_8 - 20z_3z_8 - 42z_1z_3z_8$$
$$+ 13z_2z_3z_8 + 6z_3^2z_8 + 25z_4z_8 - 12z_1z_4z_8 + 24z_5z_8 + 29z_1z_5z_8 + z_2z_5z_8 + 16z_6z_8$$

$$
\begin{aligned}
&- 26z_1z_6z_8 - 2z_1^2z_6z_8 - 8z_3z_6z_8 + 4z_6^2z_8 - 60z_7z_8 + 102z_1z_7z_8 - 14z_1^2z_7z_8\\
&+ 77z_2z_7z_8 + z_1z_2z_7z_8 + 44z_3z_7z_8 + 10z_5z_7z_8 - 41z_6z_7z_8 - 71z_7^2z_8 - 2z_1z_7^2z_8\\
&- 74z_8^2 + 63z_1z_8^2 + 8z_1^2z_8^2 - 12z_1^3z_8^2 + 55z_2z_8^2 + 3z_1z_2z_8^2 - 7z_2^2z_8^2 - 51z_3z_8^2\\
&+ 24z_1z_3z_8^2 - 2z_4z_8^2 + 17z_5z_8^2 - 40z_6z_8^2 - 8z_1z_6z_8^2 - 114z_7z_8^2 + 59z_1z_7z_8^2\\
&+ 6z_2z_7z_8^2 + 20z_7^2z_8^2 + 21z_8^3 - 86z_1z_8^3 + 18z_1^2z_8^3 - 24z_2z_8^3 - 12z_3z_8^3 - 2z_6z_8^3\\
&+ 71z_7z_8^3 + 36z_8^4 - 12z_1z_8^4 - 20z_8^5\big),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{a}_{55}(z) = -4\big(&31 + 5z_1 + 13z_1^2 - 6z_1^3 + 3z_1^4 - 2z_2 - 9z_1z_2 + 2z_2^2 + 4z_2^3 - 28z_3 + 12z_1z_3\\
&- 16z_1^2z_3 + 2z_2z_3 + 4z_1z_2z_3 + 10z_3^2 - 19z_4 + 16z_1z_4 - 12z_2z_4 - 7z_5 + 2z_1z_5 - 4z_1^2z_5\\
&- 20z_2z_5 + 4z_3z_5 - 10z_5^2 - 66z_6 - 7z_1z_6 - 9z_1^2z_6 - 6z_2z_6 - z_1z_2z_6 + 40z_3z_6 + z_4z_6\\
&- z_5z_6 + 51z_6^2 + 4z_1z_6^2 + 53z_7 + 10z_1z_7 + 6z_1^2z_7 - 6z_1^3z_7 + 16z_2z_7 - 9z_1z_2z_7 - 8z_2^2z_7\\
&- 36z_3z_7 + 12z_1z_3z_7 + 2z_2z_3z_7 + z_4z_7 + 13z_5z_7 - 2z_1z_5z_7 - 53z_6z_7 - 3z_1z_6z_7\\
&+ 6z_2z_6z_7 + 13z_7^2 + 5z_1z_7^2 + 3z_1^2z_7^2 + 18z_2z_7^2 - 8z_3z_7^2 - 7z_6z_7^2 - 9z_7^3 + 41z_8\\
&- 39z_1z_8 + 5z_1^2z_8 + 7z_1^3z_8 + 12z_2z_8 + 11z_1z_2z_8 - 10z_1^2z_2z_8 + 3z_1z_2^2z_8 - 9z_3z_8\\
&+ 3z_1z_3z_8 + 9z_2z_3z_8 + 3z_3^2z_8 - 14z_4z_8 - 6z_1z_4z_8 - 24z_5z_8 + 10z_1z_5z_8 - 3z_2z_5z_8\\
&- 31z_6z_8 - 25z_1z_6z_8 - 6z_1^2z_6z_8 + 29z_2z_6z_8 + 12z_3z_6z_8 + 12z_6^2z_8 + 62z_7z_8\\
&- 20z_1z_7z_8 + 18z_1^2z_7z_8 + 12z_2z_7z_8 - 2z_1z_2z_7z_8 - 29z_3z_7z_8 - 5z_5z_7z_8 - 57z_6z_7z_8\\
&+ 21z_7^2z_8 - z_1z_7^2z_8 - 18z_8^2 + 29z_1z_8^2 - 26z_1^2z_8^2 - 8z_2z_8^2 + z_1z_2z_8^2 + 32z_3z_8^2 - 6z_1z_3z_8^2\\
&+ 12z_4z_8^2 + 10z_5z_8^2 + 94z_6z_8^2 - 24z_7z_8^2 + 23z_1z_7z_8^2 - 10z_2z_7z_8^2 + 4z_7^2z_8^2 - 55z_8^3\\
&+ 7z_1z_8^3 + 3z_1^2z_8^3 + 9z_2z_8^3 + 6z_3z_8^3 + 13z_6z_8^3 - 64z_7z_8^3 + 20z_8^4 - 12z_1z_8^4 + 15z_8^5\big),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{a}_{34}(z) = 4\big(&-16z_1 + 40z_1^2 + 63z_1^3 + 7z_1^4 - 11z_2 + 38z_1z_2 - 7z_1^3z_2 + 23z_2^2 + 12z_1z_2^2 - 27z_3\\
&- 79z_1z_3 + 7z_1^2z_3 - z_2z_3 + 6z_1z_2z_3 - 15z_3^2 + z_4 - 46z_1z_4 - 8z_2z_4 + 20z_3z_4 + 28z_5\\
&+ 68z_1z_5 - 18z_1^2z_5 - 5z_2z_5 - 3z_1z_2z_5 + 29z_3z_5 - 5z_5^2 - 4z_6 - 66z_1z_6 - 29z_1^2z_6\\
&- 48z_2z_6 + 23z_1z_2z_6 - 5z_2^2z_6 + 9z_3z_6 - 4z_1z_3z_6 + 10z_4z_6 - 30z_5z_6 + 15z_6^2 + 4z_1z_6^2\\
&+ 42z_7 - 21z_1z_7 + 26z_1^2z_7 + 24z_1^3z_7 - 10z_2z_7 + 58z_1z_2z_7 - 5z_1^2z_2z_7 - 17z_2^2z_7 + 15z_3z_7\\
&- 30z_1z_3z_7 - z_2z_3z_7 + 34z_4z_7 - 16z_5z_7 + 67z_6z_7 - 12z_1z_6z_7 + 4z_2z_6z_7 - 16z_7^2\\
&- 45z_1z_7^2 + 5z_1^2z_7^2 + 25z_2z_7^2 + 9z_3z_7^2 - 9z_6z_7^2 - 18z_7^3 + 23z_8 + 61z_1z_8 - 24z_1^2z_8\\
&- 26z_1^3z_8 + 62z_2z_8 - 4z_1z_2z_8 - 4z_1^2z_2z_8 - 15z_2^2z_8 - 7z_1z_2^2z_8 - 23z_3z_8 - 17z_1z_3z_8\\
&- 6z_1^2z_3z_8 + z_2z_3z_8 + 6z_3^2z_8 + 25z_4z_8 + 6z_1z_4z_8 + 12z_5z_8 + 43z_1z_5z_8 + 2z_2z_5z_8\\
&- z_6z_8 - 13z_1z_6z_8 + 20z_1^2z_6z_8 - 9z_2z_6z_8 - 24z_3z_6z_8 + 4z_6^2z_8 - 109z_7z_8 + 54z_1z_7z_8\\
&- 41z_1^2z_7z_8 + 76z_2z_7z_8 - 2z_1z_2z_7z_8 + z_3z_7z_8 + 15z_5z_7z_8 - 16z_6z_7z_8 - 129z_7^2z_8\\
&- 5z_1z_7^2z_8 - 30z_8^2 - 41z_1z_8^2 - 6z_1^2z_8^2 + 6z_1^3z_8^2 + 37z_2z_8^2 - z_1z_2z_8^2 + 9z_2^2z_8^2 - 11z_3z_8^2\\
&+ 6z_1z_3z_8^2 - 26z_4z_8^2 + 15z_5z_8^2 - 38z_6z_8^2 - 32z_1z_6z_8^2 - 78z_7z_8^2 + 49z_1z_7z_8^2 - 3z_2z_7z_8^2\\
&+ 20z_7^2z_8^2 + 17z_8^3 - 4z_1z_8^3 - 6z_1^2z_8^3 - 40z_2z_8^3 + 20z_3z_8^3 - 2z_6z_8^3 + 129z_7z_8^3 + 24z_8^4\\
&- 12z_1z_8^4 - 20z_8^5\big),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{a}_{45}(z) = -4\big(&25z_1^2 + 25z_1^3 + 13z_1z_2 - 28z_1^2z_2 - 8z_1^3z_2 - 22z_2^2 - 2z_1z_2^2 - 24z_3 - 52z_1z_3\\
&+ 20z_2z_3 + 15z_1z_2z_3 + 9z_2^2z_3 - 2z_3^2 + 49z_4 + 4z_1z_4 + 9z_1^2z_4 - 6z_2z_4 - 18z_3z_4 - 45z_5\\
&+ 31z_1z_5 - 5z_1^2z_5 + 2z_2z_5 - 11z_1z_2z_5 + 4z_3z_5 - 24z_4z_5 + 9z_5^2 + 38z_6 - z_1z_6 + 12z_1^2z_6\\
&- 3z_1^3z_6 + 12z_1z_2z_6 + 9z_2^2z_6 + z_3z_6 + 13z_1z_3z_6 + 3z_2z_3z_6 - 35z_4z_6 + 59z_5z_6 + 4z_1z_5z_6
\end{aligned}
$$

$$
\begin{aligned}
&- 3z_6^2 - 14z_1z_6^2 + 5z_2z_6^2 - 11z_7 - 26z_1z_7 + 29z_1^2z_7 + 19z_1^3z_7 + 28z_2z_7 + 3z_1z_2z_7\\
&- 13z_1^2z_2z_7 - 22z_2^2z_7 + 5z_1z_2^2z_7 - 20z_3z_7 - 33z_1z_3z_7 + 5z_2z_3z_7 + 5z_3^2z_7 + 38z_4z_7\\
&- 10z_1z_4z_7 - 24z_5z_7 + 15z_1z_5z_7 + z_2z_5z_7 - 3z_6z_7 - 4z_1^2z_6z_7 + 31z_2z_6z_7 - 5z_6^2z_7\\
&- 42z_7^2 + 3z_1z_7^2 + 4z_1^2z_7^2 + 28z_2z_7^2 - 4z_1z_2z_7^2 + 4z_3z_7^2 - 9z_5z_7^2 - 10z_6z_7^2 - 31z_7^3\\
&+ 9z_1z_7^3 - 5z_8 - 11z_1z_8 - 4z_1^2z_8 - 12z_1^3z_8 + 6z_1^4z_8 + 14z_2z_8 - 4z_1z_2z_8 + 3z_1^2z_2z_8\\
&- 2z_2^2z_8 - z_1z_2^2z_8 + 7z_2^3z_8 - 39z_3z_8 + 40z_1z_3z_8 - 31z_1^2z_3z_8 + 26z_2z_3z_8 + 7z_1z_2z_3z_8\\
&+ 19z_3^2z_8 - 71z_4z_8 + 15z_1z_4z_8 - 21z_2z_4z_8 - 42z_5z_8 - 13z_1z_5z_8 - 7z_1^2z_5z_8 + 5z_2z_5z_8\\
&+ 13z_3z_5z_8 - 73z_6z_8 - 50z_1z_6z_8 - 16z_1^2z_6z_8 - 42z_2z_6z_8 - 7z_1z_2z_6z_8 + 50z_3z_6z_8\\
&+ 19z_5z_6z_8 + 7z_6^2z_8 - 22z_7z_8 + 53z_1z_7z_8 - 30z_1^2z_7z_8 - 6z_1^3z_7z_8 + 44z_2z_7z_8\\
&- 5z_1z_2z_7z_8 - 7z_2^2z_7z_8 - 53z_3z_7z_8 + z_1z_3z_7z_8 + 19z_4z_7z_8 - 52z_5z_7z_8 + 43z_6z_7z_8\\
&- 10z_1z_6z_7z_8 - 37z_7^2z_8 + 17z_1z_7^2z_8 + 2z_2z_7^2z_8 + 129z_8^2 - 42z_1z_8^2 - 21z_1^2z_8^2 + 7z_1^3z_8^2\\
&- 7z_2z_8^2 + 32z_1z_2z_8^2 - 13z_1^2z_2z_8^2 + 2z_2^2z_8^2 + 11z_3z_8^2 + 6z_1z_3z_8^2 + 12z_2z_3z_8^2 - 25z_4z_8^2\\
&+ 28z_5z_8^2 + 7z_1z_5z_8^2 - 28z_6z_8^2 + 16z_1z_6z_8^2 + 3z_2z_6z_8^2 + 103z_7z_8^2 - 30z_1z_7z_8^2 + 24z_1^2z_7z_8^2\\
&+ 53z_2z_7z_8^2 - 14z_3z_7z_8^2 - 14z_7^2z_8^2 - 37z_8^3 + 49z_1z_8^3 - 23z_1^2z_8^3 - 8z_2z_8^3 + 7z_1z_2z_8^3\\
&+ 38z_3z_8^3 + 28z_5z_8^3 + 12z_6z_8^3 + 31z_7z_8^3 - 21z_1z_7z_8^3 - 52z_8^4 + 3z_1z_8^4 - 9z_2z_8^4 + 12z_8^5\big),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{a}_{44}(z) = -4\big(&31 + 16z_1 + 18z_1^2 - z_1^3 + 5z_1^4 + 7z_1^5 + 5z_2 + 26z_1z_2 - 5z_1^2z_2 + 3z_2^2 - 30z_1z_2^2 - 5z_2^3\\
&- 11z_3 - 21z_1z_3 - 45z_1^2z_3 - 35z_1^3z_3 - 24z_2z_3 - 10z_1z_2z_3 + 35z_3^2 + 35z_1z_3^2 + 23z_4\\
&+ 35z_1z_4 + 30z_1^2z_4 + 15z_2z_4 + 5z_1z_2z_4 - 30z_3z_4 - 15z_4^2 + 36z_5 + 13z_1z_5 + 3z_1^2z_5\\
&+ 4z_1^3z_5 - 5z_2z_5 - 19z_1z_2z_5 + 3z_2^2z_5 - 16z_3z_5 - 6z_1z_3z_5 + z_2z_3z_5 - 12z_4z_5 + 31z_5^2\\
&+ 2z_1z_5^2 - 11z_6 + 20z_1z_6 + 51z_1^2z_6 + 16z_1^3z_6 - 3z_2z_6 + 34z_1z_2z_6 - 7z_1^2z_2z_6 + 19z_2^2z_6\\
&+ 2z_1z_2^2z_6 - 12z_3z_6 - 9z_1z_3z_6 + 5z_2z_3z_6 + 2z_3^2z_6 - 24z_4z_6 - 4z_1z_4z_6 - 13z_5z_6\\
&+ 2z_1z_5z_6 + 3z_2z_5z_6 - 2z_6^2 - 22z_1z_6^2 + 2z_2z_6^2 - 4z_3z_6^2 + 2z_6^3 + 11z_7 - 35z_1z_7 - 4z_1^2z_7\\
&+ 3z_1^3z_7 - 3z_1^4z_7 + 15z_2z_7 + 16z_1z_2z_7 - 11z_1^2z_2z_7 - 15z_2^2z_7 + 4z_1z_2^2z_7 + 3z_2^3z_7 + 7z_3z_7\\
&- 21z_1z_3z_7 + 9z_1^2z_3z_7 - 11z_2z_3z_7 + 3z_1z_2z_3z_7 - 3z_3^2z_7 + 31z_4z_7 - 17z_1z_4z_7\\
&- 9z_2z_4z_7 + 13z_5z_7 + 34z_1z_5z_7 - 4z_1^2z_5z_7 + 22z_2z_5z_7 + 9z_3z_5z_7 - 7z_6z_7 - 26z_1z_6z_7\\
&+ z_1^2z_6z_7 - 17z_2z_6z_7 - 7z_1z_2z_6z_7 - 9z_5z_6z_7 + 4z_6^2z_7 - 18z_7^2 - 28z_1z_7^2 - 5z_1^2z_7^2\\
&+ 2z_1^3z_7^2 + 12z_2z_7^2 - 2z_1z_2z_7^2 + 35z_3z_7^2 - 9z_1z_3z_7^2 + 9z_4z_7^2 - 18z_5z_7^2 + 9z_6z_7^2\\
&+ 9z_1z_6z_7^2 - 28z_7^3 - 15z_8 - 36z_1z_8 - 33z_1^2z_8 + 30z_1^3z_8 + 16z_2z_8 - 23z_1z_2z_8\\
&+ 37z_1^2z_2z_8 - 3z_1^3z_2z_8 - 22z_2^2z_8 + 18z_1z_2^2z_8 - 5z_2^3z_8 - 15z_3z_8 - 37z_1z_3z_8 - 45z_2z_3z_8\\
&+ 5z_1z_2z_3z_8 + 4z_2^2z_3z_8 + 38z_3^2z_8 + 73z_4z_8 - 41z_1z_4z_8 + 4z_1^2z_4z_8 + 9z_2z_4z_8 - 8z_3z_4z_8\\
&- 32z_5z_8 - 28z_1^2z_5z_8 - 7z_2z_5z_8 - 2z_1z_2z_5z_8 + 18z_3z_5z_8 + 12z_5^2z_8 + 4z_6z_8 + 13z_1z_6z_8\\
&- 28z_1^2z_6z_8 - 4z_1^3z_6z_8 + 18z_2z_6z_8 + 19z_1z_2z_6z_8 - 3z_2^2z_6z_8 + 8z_3z_6z_8 + 8z_1z_3z_6z_8\\
&- 6z_4z_6z_8 - 4z_5z_6z_8 + 4z_6^2z_8 - 4z_1z_6^2z_8 - 33z_7z_8 - 6z_1z_7z_8 - 45z_1^2z_7z_8 + 3z_1^3z_7z_8\\
&+ 50z_2z_7z_8 - 41z_1z_2z_7z_8 - 9z_1^2z_2z_7z_8 - 13z_2^2z_7z_8 + 46z_3z_7z_8 + 9z_1z_3z_7z_8\\
&+ 8z_2z_3z_7z_8 + 4z_4z_7z_8 + 11z_5z_7z_8 - 6z_1z_5z_7z_8 - 49z_6z_7z_8 + 5z_1z_6z_7z_8 + 6z_2z_6z_7z_8\\
&- 85z_7^2z_8 + 96z_1z_7^2z_8 + 3z_1^2z_7^2z_8 + 26z_2z_7^2z_8 - 9z_6z_7^2z_8 - 9z_7^3z_8 - 42z_8^2 + 25z_1z_8^2\\
&+ 11z_1^2z_8^2 - 44z_1^3z_8^2 + 6z_1^4z_8^2 + 37z_2z_8^2 - 51z_1z_2z_8^2 + 4z_1^2z_2z_8^2 + 11z_2^2z_8^2 - 38z_3z_8^2\\
&+ 89z_1z_3z_8^2 - 24z_1^2z_3z_8^2 + 12z_3^2z_8^2 - 48z_4z_8^2 + 16z_1z_4z_8^2 - 63z_5z_8^2 + 2z_1z_5z_8^2\\
&- 11z_2z_5z_8^2 - 22z_6z_8^2 - 5z_1z_6z_8^2 + 14z_1^2z_6z_8^2 + z_2z_6z_8^2 + 4z_3z_6z_8^2 + 2z_6^2z_8^2 - 41z_7z_8^2
\end{aligned}
$$

$$+ 74z_1z_7z_8^2 - 18z_1^2z_7z_8^2 - 54z_2z_7z_8^2 - z_1z_2z_7z_8^2 - 13z_3z_7z_8^2 + 21z_5z_7z_8^2 - 4z_6z_7z_8^2 + 94z_7^2z_8^2 - 17z_1z_7^2z_8^2 + 75z_8^3 - 64z_1z_8^3 + 32z_1^2z_8^3 - 44z_2z_8^3 + 26z_1z_2z_8^3 + 4z_2^2z_8^3 + 6z_3z_8^3 - 8z_4z_8^3 + 14z_5z_8^3 + 34z_6z_8^3 + 4z_1z_6z_8^3 + 95z_7z_8^3 - 64z_1z_7z_8^3 - z_2z_7z_8^3 + 9z_7^2z_8^3 + 7z_8^4 + 8z_1z_8^4 - 8z_1^2z_8^4 + 12z_2z_8^4 + 16z_3z_8^4 - 4z_6z_8^4 - 42z_7z_8^4 - 32z_8^5 + 16z_1z_8^5 + 10z_8^6\big).$$

To fully determine the operator Δ^κ, we have to find the remaining b-coefficients; these can be reached, as we have pointed out at the end of section 2, once we know the b^1- and $\boldsymbol{a}$-coefficients, and the necessary monomials M_λ. Through a recursive procedure, starting from the outer dots in the Dynkin diagram, it is possible to obtain all these monomials by using, in each step, only the part of the Hamiltonian which involves the b's associated with the dots already computed. In such a way, we finally obtain

$$\begin{aligned}
b_8(z) &= 4(8 + 29z_8),\\
b_1(z) &= 4(14 + 46z_1 + 7z_8),\\
b_7(z) &= 4(7 + 7z_1 + 57z_7 + 21z_8),\\
b_2(z) &= 4(-8 + 16z_1 + 68z_2 + 6z_7 + 11z_8),\\
b_6(z) &= 4\big(1 + z_1 + 8z_2 + 84z_6 - 3z_7 - 28z_8 + 6z_1z_8 + 12z_8^2\big),\\
b_3(z) &= 4\big(9 - 7z_1 + 91z_3 + 5z_6 + 5z_7 - 12z_8 + 14z_1z_8 - 2z_8^2\big),\\
b_5(z) &= 4\big(6 + 5z_1 + 7z_1^2 + 26z_2 - 7z_3 + 110z_5 + z_6 - 14z_7 + 5z_1z_7 + 32z_8\\
&\quad - 19z_1z_8 + 7z_2z_8 + z_7z_8 - 13z_8^2\big),\\
b_4(z) &= 4\big(17 - 20z_1 - 21z_1^2 - 2z_2 + 8z_1z_2 + 22z_3 + 135z_4 - 10z_5 + 40z_6 + 4z_1z_6 - 28z_7\\
&\quad - 6z_1z_7 + 6z_2z_7 - 9z_7^2 - 94z_8 + 28z_1z_8 + 6z_1^2z_8 + 22z_2z_8 - 6z_3z_8 + 4z_6z_8 - 48z_7z_8\\
&\quad + 51z_8^2 - 18z_1z_8^2 + 18z_8^3\big).
\end{aligned}$$

In summary, the explicit form of the operator Δ^κ in terms of the z-variables is achieved by substituting in (12) the coefficients $\boldsymbol{a}_{jk}, b_j^1$ and b_j obtained above. With this operator we solve the eigenvalue problem (6) and finally we get the quantum wavefunctions $\Psi_{\mathbf{m}}^\kappa$ by means of (5), and the energy levels $E_{\mathbf{m}}(\kappa)$ by using (8).

Remark. Once the operator Δ^κ is known, we have then developed several recursive methods to find some other results such as higher order polynomials, the deformed Clebsch–Gordan series, etc (see [14]). For instance, for the case of E_8 we are dealing with and the particular value $\kappa = 1$, we have used these methods to compute the full list of second-order characters $\chi_{\lambda_j+\lambda_k}(z)$ and also some other of higher order (see [20]). As an illustration, we show here only a few of the simplest polynomials $\Phi_{\mathbf{m}}^\kappa$ which solve the eigenvalue problem (6):

$$\begin{aligned}
\Phi^\kappa_{00000001} &= z_8 + \frac{8(\kappa-1)}{1+29\kappa}\\
\Phi^\kappa_{10000000} &= z_1 + \frac{7(\kappa-1)}{1+17\kappa}z_8 + \frac{21(1-8\kappa+7\kappa^2)}{(1+17\kappa)(1+23\kappa)}\\
\Phi^\kappa_{00000002} &= z_8^2 - \frac{2}{1+\kappa}z_7 - \frac{14\kappa}{(1+\kappa)(1+6\kappa)}z_1 - \frac{2(3+6\kappa+263\kappa^2-48\kappa^3)}{(1+\kappa)(1+6\kappa)(3+29\kappa)}z_8\\
&\quad - \frac{2(24+462\kappa+679\kappa^2+5971\kappa^3-192\kappa^4)}{(1+\kappa)(1+6\kappa)(2+29\kappa)(3+29\kappa)}\\
\Phi^\kappa_{00000010} &= z_7 + \frac{7(\kappa-1)}{1+11\kappa}z_1 + \frac{14(1-11\kappa+10\kappa^2)}{(1+11\kappa)(1+14\kappa)}z_8 - \frac{7(1-112\kappa+281\kappa^2-170\kappa^3)}{(1+11\kappa)(1+14\kappa)(1+19\kappa)}
\end{aligned}$$

$$\Phi^{\kappa}_{01000000} = z_2 + \frac{6(\kappa - 1)}{1 + 11\kappa} z_7 + \frac{(13 - 122\kappa + 109\kappa^2)}{(1 + 11\kappa)^2} z_1 + \frac{8(1 + 77\kappa - 223\kappa^2 + 145\kappa^3)}{(1 + 11\kappa)^2(1 + 13\kappa)} z_8 - \frac{7(7 + 148\kappa - 1698\kappa^2 + 2348\kappa^3 - 805\kappa^4)}{(1 + 11\kappa)^2(1 + 13\kappa)(1 + 17\kappa)}.$$

Acknowledgments

This paper was completed during the visit of one of the authors (AMP) to the Max Planck Institut für Gravitationsphysik, and he thanks the institute staff for the hospitality. This work was partially supported by the Spanish Government under grants MTM2006-10532 (JFN) and FIS2006-09417 (WGF and AMP).

References

[1] Calogero F 2001 *Classical Many Body Problems Amenable to Exact Treatments* (Berlin: Springer)
[2] Perelomov A M 1990 *Integrable Systems of Classical Mechanics and Lie Algebras* (Basle: Birkhäuser)
[3] Calogero F 1971 *J. Math. Phys.* **12** 419–36
[4] Sutherland B 1972 *Phys. Rev.* A **4** 2019–21
[5] Moser J 1975 *Adv. Math.* **16** 197–220
[6] Olshanetsky M A and Perelomov A M 1981 *Phys. Rep.* **71** 314–400
Olshanetsky M A and Perelomov A M 1983 *Phys. Rep.* **94** 313–404
[7] van Diejen J F and Vinet L (ed) 2000 *Calogero–Moser–Sutherland Models* (Berlin: Springer)
[8] Polychronakos A P 2006 *J. Phys. A: Math. Gen.* **39** 12793–846
[9] Iso S and Rey S J 1995 *Phys. Lett.* B **352** 111–6
[10] Caselle M 1995 *Phys. Rev. Lett.* **74** 2776
[11] D'Hoker E and Pong D H 1999 arXiv:hep-th/9912271
[12] Gibbons G W and Townsend P K 1999 *J. Phys. Lett.* B **454** 187–92
[13] Perelomov A M 1998 *J. Phys. A: Math. Gen.* **31** L31–7
Perelomov A M, Ragoucy E and Zaugg Ph 1998 *J. Phys. A: Math. Gen.* **31** L559–65
Perelomov A M 1999 *J. Phys. A: Math. Gen.* **32** 8563–76
García Fuertes W, Lorente M and Perelomov A M 2001 *J. Phys. A: Math. Gen.* **34** 10963–73
[14] Fernández Núñez J, García Fuertes W and Perelomov A M 2003 *J. Math. Phys.* **44** 4957–74
Fernández Núñez J, García Fuertes W and Perelomov A M 2005 *J. Nonlinear Math. Phys.* **12** (Suppl. 1) 280–301
Fernández Núñez J, García Fuertes W and Perelomov A M 2005 *J. Math. Phys.* **46** 073508
Fernández Núñez J, García Fuertes W and Perelomov A M 2005 *J. Math. Phys.* **46** 103505
Fernández Núñez J, García Fuertes W and Perelomov A M 2008 *Theor. Math. Phys.* **154** 240–9
[15] Boya L J 2003 arXiv:hep-th/0301037
Ramond P 2003 arXiv:hep-th/0301050
[16] Onishchik A L and Vinberg E B 1990 *Lie Groups and Algebraic Groups* (Berlin: Springer)
[17] Cornwell J F 1984 *Group Theory in Physics* vol 2 (New York: Academic)
Bourbaki N 1975 *Groupes et Algebres de Lie* (Paris: Hermann)
[18] Khastgir S P, Pocklington A J and Sasaki R 2000 *J. Phys. A: Math. Gen.* **33** 9033–64
[19] 1996 LiE *A Computer Algebra Package for Lie Groups Computations* www-math.univ-poitiers.fr/~maavl/LiE/
[20] Fernández Núñez J, García Fuertes W and Perelomov A M 2008 arXiv:0806.1011